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A counterexample to the Fourteenth Problem of Hilbert in dimension four

Shigeru Kuroda [1]

Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan

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Abstract

This paper presents a family of new counterexamples to Hilbert's Fourteenth Problem. They are realized as subfields of the rational function field of four variables over a field of characteristic zero of transcendence degree three. It was previously not known whether any counterexample could be found as a subfield of a rational function field of four variables, or whether counterexamples with minimal transcendence degree three could exist.

1. Introduction

Let K be a field of characteristic zero, $K[\mathbf{x}] = K[x_1, \ldots, x_n]$ the polynomial ring in n variables over K, and $K(\mathbf{x})$ its field of fractions. Then, the Fourteenth Problem of Hilbert asks whether the K-subalgebra $L \cap K[\mathbf{x}]$ of $K[\mathbf{x}]$ is finitely generated for a subfield L of $K(\mathbf{x})$ containing K. The first counterexample to this problem was found by Nagata [12] in 1958 in the case where $n = 2s^2$ for each integer $s \geqslant 4$. In 1990, Roberts [14] constructed a new counterexample when $n = 7$. Kojima and Miyanishi [6] gave a similar counterexample for each odd number $n \geqslant 7$. Recently, we generalized Roberts' example further in [8], and showed a detailed sufficient condition for certain invariant subrings of $K[\mathbf{x}]$ not to be

E-mail address: kuroda@kurims.kyoto-u.ac.jp.
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finitely generated in the case where $n \geqslant 7$. In lower dimensions, Freudenburg [4] gave a counterexample for $n = 6$, and Daigle and Freudenburg [1] gave one for $n = 5$.

While there exists a counterexample for each $n \geqslant 5$, the answer to the Fourteenth Problem of Hilbert is known to be affirmative for $n \leqslant 2$. Actually, Zariski [15] showed that $L \cap K[\mathbf{x}]$ is finitely generated if the transcendence degree of L over K is at most two. In case of $n = 3, 4$, there were only positive solutions for some special cases (cf. [2,7,9,11]).

In the case of Nagata's counterexample, the transcendence degree of L over K is four. After giving the counterexample, Nagata [13] proposed two open problems concerning the Fourteenth Problem of Hilbert. His second problem asks the answer to the Fourteenth Problem in the case where the transcendence degree of L over K is three.

In the present paper, we give the first counterexample to the Fourteenth Problem of Hilbert for $n = 4$. We also settle Nagata's second problem negatively by our new counterexample. Furthermore, we show that the counterexample cannot be obtained as the kernel of any locally nilpotent derivation. Here, a K-linear map $D : A \to A$ of a commutative K-algebra A is called a *derivation* if $D(fg) = D(f)g + fD(g)$ for any $f, g \in A$. Then, the kernel

$$A^D = \{ f \in A \mid D(f) = 0 \}$$

of D is a K-subalgebra of A. A derivation D on A is said to be *locally nilpotent* if, for each $f \in A$, there exists $r > 0$ such that $D^r(f) = 0$. We note that each of the counterexamples in [1,4,6,8,14] is given as the kernel of certain locally nilpotent derivation on $K[\mathbf{x}]$.

Assume that $n = 4$. Let γ and $\delta_{i,j}$ be integers for $1 \leqslant i \leqslant 3$ and $1 \leqslant j \leqslant 4$ such that $\gamma, \delta_{i,j} \geqslant 1$ and $\delta_{i,4} \geqslant 0$ for $1 \leqslant i, j \leqslant 3$, and let $K(\Pi)$ be the subfield of $K(\mathbf{x})$ generated by

$$\Pi_1 = x_4^{\gamma} - x_1^{-\delta_{1,1}} x_2^{\delta_{1,2}} x_3^{\delta_{1,3}} x_4^{\delta_{1,4}}, \qquad \Pi_2 = x_4^{\gamma} - x_1^{\delta_{2,1}} x_2^{-\delta_{2,2}} x_3^{\delta_{2,3}} x_4^{\delta_{2,4}},$$
$$\Pi_3 = x_4^{\gamma} - x_1^{\delta_{3,1}} x_2^{\delta_{3,2}} x_3^{-\delta_{3,3}} x_4^{\delta_{3,4}} \tag{1.1}$$

over K.

The following is the main result of this paper.

Theorem 1.1. *Assume that $n = 4$. If*

$$\frac{\delta_{1,1}}{\delta_{1,1} + \min\{\delta_{2,1}, \delta_{3,1}\}} + \frac{\delta_{2,2}}{\delta_{2,2} + \min\{\delta_{3,2}, \delta_{1,2}\}} + \frac{\delta_{3,3}}{\delta_{3,3} + \min\{\delta_{1,3}, \delta_{2,3}\}} < 1, \tag{1.2}$$

then $K(\Pi) \cap K[\mathbf{x}]$ is not finitely generated over K. Moreover, $K(\Pi) \cap K[\mathbf{x}]$ is not contained in the kernel $K[\mathbf{x}]^D$ of any nonzero locally nilpotent derivation D on $K[\mathbf{x}]$.

Clearly, the transcendence degree of $K(\Pi)$ over K is at most three. Hence, Theorem 1.1 implies that Nagata's second problem has a negative answer.

We remark that $K(\Pi) \cap K[\mathbf{x}]$ is not equal to the invariant subring $K[\mathbf{x}]^G$ of $K[\mathbf{x}]$ for $n = 4$ for any algebraic action of an algebraic group G by the following reason, which was pointed out by the referee. Without loss of generality, we may assume that G acts on $K[\mathbf{x}]$ faithfully. Suppose to the contrary that $K(\Pi) \cap K[\mathbf{x}] = K[\mathbf{x}]^G$. Then, G is not reductive by the Hilbert finiteness theorem. Hence, G contains the one-dimensional additive group G_a as a subgroup, and so $K(\Pi) \cap K[\mathbf{x}] \subset K[\mathbf{x}]^{G_a}$. This contradicts Theorem 1.1, since $K[\mathbf{x}]^{G_a}$ is equal to the kernel $K[\mathbf{x}]^D$ of a nonzero locally nilpotent derivation D on $K[\mathbf{x}]$.

Note: the author recently constructed a counterexample for $n = 3$ [10]. Hence, Hilbert's Fourteenth Problem is settled for every n at last.

2. The structure of $K(\Pi) \cap K[\mathbf{x}]$

Let $K[\mathbf{y}]$ and $K[\mathbf{y}, \mathbf{y}^{-1}]$ be the polynomial ring and the Laurent polynomial ring in y_1, y_2, y_3, y_4 over K, respectively, and $K(\mathbf{y})$ the field of fractions of $K[\mathbf{y}]$. For $a = (a_1, a_2, a_3, a_4) \in \mathbf{Z}^4$, we denote $\mathbf{x}^a = x_1^{a_1} x_2^{a_2} x_3^{a_3} x_4^{a_4}$ and $\mathbf{y}^a = y_1^{a_1} y_2^{a_2} y_3^{a_3} y_4^{a_4}$. If f is the Laurent polynomial $\sum_{a \in \mathbf{Z}^4} \lambda_a \mathbf{x}^a$ or $\sum_{a \in \mathbf{Z}^4} \lambda_a \mathbf{y}^a$, then define the *support* $\operatorname{supp}(f)$ of f by

$$\operatorname{supp}(f) = \left\{ a \in \mathbf{Z}^4 \;\middle|\; \lambda_a \neq 0 \right\}.$$

We denote by $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$ the coordinate unit vectors of $\mathbf{R}^4$, and by $\mathbf{Z}_{\geqslant 0}$ the set of nonnegative integers.

Let $K[\mathbf{x}]' = K[\mathbf{x}^{\pm\delta_1}, \mathbf{x}^{\pm\delta_2}, \mathbf{x}^{\pm\delta_3}, x_4^{\pm\gamma}]$, and $K(\mathbf{x})'$ its field of fractions, where

$$\delta_1 = (-\delta_{1,1}, \delta_{1,2}, \delta_{1,3}, \delta_{1,4}), \qquad \delta_2 = (\delta_{2,1}, -\delta_{2,2}, \delta_{2,3}, \delta_{2,4}),$$
$$\delta_3 = (\delta_{3,1}, \delta_{3,2}, -\delta_{3,3}, \delta_{3,4}) \tag{2.1}$$

and $\delta_{i,j}$ and γ are as defined above. Then, we may define an isomorphism $\Phi : K[\mathbf{y}, \mathbf{y}^{-1}] \to K[\mathbf{x}]'$ of K-algebras by $y_i \mapsto \mathbf{x}^{\delta_i}$ for $i = 1, 2, 3$ and $y_4 \mapsto x_4^{\gamma}$. To see this, it suffices to verify that $\delta_1, \delta_2, \delta_3$ and $\gamma \mathbf{e}_4$ are linearly independent over $\mathbf{R}$. Suppose the contrary. Then, there exists $(\lambda_1, \lambda_2, \lambda_3) \in \mathbf{R}^3 \setminus \{0\}$ such that the first three components of $\sum_{i=1}^{3} \lambda_i \delta_i$ are zero. It implies that $\lambda_1, \lambda_2, \lambda_3 > 0$ or $\lambda_1, \lambda_2, \lambda_3 < 0$. So, without loss of generality, we may assume that $0 < \lambda_1 \leqslant \lambda_2 \leqslant \lambda_3$. Since $\delta_{1,1} = (\lambda_2/\lambda_1)\delta_{2,1} + (\lambda_3/\lambda_1)\delta_{3,1}$, we have $\delta_{1,1} \geqslant 2 \min\{\delta_{2,1}, \delta_{3,1}\}$. Hence, $\xi_1 \geqslant 2/3$. Here, we set

$$\xi_1 = \frac{\delta_{1,1}}{\delta_{1,1} + \min\{\delta_{2,1}, \delta_{3,1}\}}, \qquad \xi_2 = \frac{\delta_{2,2}}{\delta_{2,2} + \min\{\delta_{3,2}, \delta_{1,2}\}},$$
$$\xi_3 = \frac{\delta_{3,3}}{\delta_{3,3} + \min\{\delta_{1,3}, \delta_{2,3}\}}. \tag{2.2}$$

Since $\delta_{2,2} = (\lambda_1/\lambda_2)\delta_{1,2} + (\lambda_3/\lambda_2)\delta_{3,2}$, we have $\delta_{2,2} > \min\{\delta_{3,2}, \delta_{1,2}\}$. Hence, $\xi_2 > 1/2$. This contradicts (1.2). Therefore, Φ is an isomorphism, and hence it induces an isomorphism $K(\mathbf{y}) \to K(\mathbf{x})'$.

Consider the derivation E on $K[\mathbf{y}]$ defined by $E(y_i) = 1$ for each i. It is naturally extended to derivations on $K[\mathbf{y}, \mathbf{y}^{-1}]$ and $K(\mathbf{y})$, respectively. Since $E(y_4 - y_i) = 0$ and $\Pi_i = \Phi(y_4 - y_i)$ for $i = 1, 2, 3$, we have $K(\Pi) \subset \Phi(K(\mathbf{y})^E)$.

Lemma 2.1. *It follows that* $K[\mathbf{y},\mathbf{y}^{-1}]^E = K[\mathbf{y}]^E = K[y_4-y_1, y_4-y_2, y_4-y_3]$.

Proof. Consider the automorphism $\sigma : K[\mathbf{y}] \to K[\mathbf{y}]$ defined by $y_i \mapsto y_4 - y_i$ for $i = 1,2,3$ and $y_4 \mapsto y_4$. Then, $\sigma^{-1} \circ E \circ \sigma$ is equal to the partial derivative $\partial/\partial y_4$. Hence,

$$K[\mathbf{y}]^E = \sigma\bigl(K[\mathbf{y}]^{\sigma^{-1}\circ E\circ\sigma}\bigr) = \sigma\bigl(K[y_1,y_2,y_3]\bigr) = K[y_4-y_1, y_4-y_2, y_4-y_3].$$

Now, we show $K[\mathbf{y},\mathbf{y}^{-1}]^E = K[\mathbf{y}]^E$. It is clear that $K[\mathbf{y},\mathbf{y}^{-1}]^E \supset K[\mathbf{y}]^E$. For the converse, suppose that $E(f/\mathbf{y}^a) = 0$ for some $f \in K[\mathbf{y}] \setminus \{0\}$ and $a \in (\mathbf{Z}_{\geqslant 0})^4 \setminus \{0\}$ such that $\gcd(f, \mathbf{y}^a) = 1$. Then,

$$E(f) = E\bigl((f/\mathbf{y}^a)\mathbf{y}^a\bigr) = E\bigl(f/\mathbf{y}^a\bigr)\mathbf{y}^a + \bigl(f/\mathbf{y}^a\bigr)E\bigl(\mathbf{y}^a\bigr) = \bigl(f/\mathbf{y}^a\bigr)E\bigl(\mathbf{y}^a\bigr),$$

which is in $K[\mathbf{y}]$. Hence, $E(\mathbf{y}^a) = h\mathbf{y}^a$ for some $h \in K[\mathbf{y}]$ by the assumption that $\gcd(f,\mathbf{y}^a) = 1$. Note that E is a locally nilpotent derivation on $K[\mathbf{y}]$, and an eigenvalue of a nonzero locally nilpotent derivation must be zero (see [3, Proposition 1.3.32(ii)]). Hence, we have $E(\mathbf{y}^a) = 0$. It implies that $y_i \in K[\mathbf{y}]^E$ for some i, since $K[\mathbf{y}]^E$ is factorially closed in $K[\mathbf{y}]$ (see [3, Proposition 1.3.32(iii)]) and $\mathbf{y}^a \neq 1$. This contradicts the definition of E. Therefore, $K[\mathbf{y},\mathbf{y}^{-1}]^E \subset K[\mathbf{y}]^E$. □

Let us denote $K[\Pi] = K[\Pi_1, \Pi_2, \Pi_3]$. By Lemma 2.1, $K[\mathbf{y}]^E$ is isomorphic to $K[\Pi]$ via Φ; in fact, $K[\Pi] = \Phi(K[\mathbf{y}]^E)$.

Lemma 2.2. *It follows that* $K(\Pi) \cap K[\mathbf{x}] = K[\Pi] \cap K[\mathbf{x}]$.

Proof. It is clear that $K(\Pi) \cap K[\mathbf{x}] \supset K[\Pi] \cap K[\mathbf{x}]$. We show that $K(\Pi) \cap K[\mathbf{x}] \subset K[\mathbf{x}]'$ in the following paragraph. Then, $K(\Pi) \cap K[\mathbf{x}] \subset K[\Pi] \cap K[\mathbf{x}]$ is proved as follows. By Lemma 2.1, we have

$$\begin{aligned}\Phi^{-1}\bigl(K(\Pi) \cap K[\mathbf{x}]'\bigr) &= \Phi^{-1}\bigl(K(\Pi)\bigr) \cap \Phi^{-1}\bigl(K[\mathbf{x}]'\bigr) \subset K(\mathbf{y})^E \cap K\bigl[\mathbf{y},\mathbf{y}^{-1}\bigr] \\ &= K\bigl[\mathbf{y},\mathbf{y}^{-1}\bigr]^E = K[\mathbf{y}]^E.\end{aligned}$$

Hence, $K(\Pi) \cap K[\mathbf{x}]' \subset K[\Pi]$, since $\Phi(K[\mathbf{y}]^E) = K[\Pi]$. By the assumption that $K(\Pi) \cap K[\mathbf{x}] \subset K[\mathbf{x}]'$, we have $K(\Pi) \cap K[\mathbf{x}] \subset K[\Pi]$. Thus, we get $K(\Pi) \cap K[\mathbf{x}] \subset K[\Pi] \cap K[\mathbf{x}]$.

Since $K(\Pi) \cap K[\mathbf{x}] \subset K(\mathbf{x})' \cap K[\mathbf{x},\mathbf{x}^{-1}]$, we show that $K(\mathbf{x})' \cap K[\mathbf{x},\mathbf{x}^{-1}] \subset K[\mathbf{x}]'$. Suppose that there exist $F, G \in K[\mathbf{x}]'$ such that $F/G \in K[\mathbf{x},\mathbf{x}^{-1}] \setminus K[\mathbf{x}]'$. If h is the sum of terms of F/G contained in $K[\mathbf{x}]'$, then $F - Gh \in K[\mathbf{x}]'$ and $(F - Gh)/G = F/G - h \in K[\mathbf{x},\mathbf{x}^{-1}] \setminus K[\mathbf{x}]'$. So, by replacing F with $F - Gh$, we may assume that any monomial appearing in F/G is not in $K[\mathbf{x}]'$. For each $f \in K[\mathbf{x},\mathbf{x}^{-1}] \setminus \{0\}$, let $\mathrm{in}(f)$ denote the maximal monomial appearing in f with nonzero coefficient for the total ordering $\preccurlyeq$ on the set of monomials defined by $\mathbf{x}^a \preccurlyeq \mathbf{x}^b$ if the first nonzero component of $b - a$ is positive. Then, $\mathrm{in}(F)/\mathrm{in}(G) \in K[\mathbf{x}]'$, since $\mathrm{in}(F), \mathrm{in}(G) \in K[\mathbf{x}]'$. This is a contradiction, since $\mathrm{in}(F)/\mathrm{in}(G) = \mathrm{in}(F/G)$, which is not in $K[\mathbf{x}]'$. Hence, $K(\mathbf{x})' \cap K[\mathbf{x},\mathbf{x}^{-1}] \subset K[\mathbf{x}]'$. □

Lemma 2.3. *If* (a_1, a_2, a_3, a_4) *is in* $\mathrm{supp}(f)$ *with* $a_4 > 0$ *for* $f \in K(\Pi) \cap K[\mathbf{x}]$, *then* $a_1 + a_2 + a_3$ *is positive.*

Proof. Suppose to the contrary that $(0, 0, 0, a_4)$ is in $\mathrm{supp}(f)$ with $a_4 > 0$ for some $f \in K(\Pi) \cap K[\mathbf{x}]$. Then, $\Phi^{-1}(f)$ is in $K[\mathbf{y}]^E$, since $K(\Pi) \cap K[\mathbf{x}] = K[\Pi] \cap K[\mathbf{x}]$ by Lemma 2.2 and $\Phi^{-1}(K[\Pi]) = K[\mathbf{y}]^E$. Note that any monomial in $K[\mathbf{y}] \setminus K[y_4]$ is not sent to $x_4^{a_4}$ by Φ. Hence, $l = a_4/\gamma$ is an integer, and y_4^l appears in $\Phi^{-1}(f)$. We remark that $E(\mathbf{y}^b) = \sum_{i=1}^4 b_i \mathbf{y}^b y_i^{-1}$ for $b = (b_1, b_2, b_3, b_4) \in (\mathbf{Z}_{\geqslant 0})^4$. Hence, for any $g \in K[\mathbf{y}]^E$ and $b \in \mathrm{supp}(g)$, if a monomial $\mathbf{y}^c$ appears in $E(\mathbf{y}^b)$ with nonzero coefficient, then there exists $d \in \mathrm{supp}(g) \setminus \{b\}$ such that $\mathbf{y}^c = \mathbf{y}^d y_i^{-1}$ for some i. Actually, if such d did not exist, then $\mathbf{y}^c$ would appear in $E(g)$. So, there appears in $\Phi^{-1}(f)$ a monomial $\mathbf{y}^d \neq y_4^l$ such that $y_4^{l-1} = \mathbf{y}^d y_i^{-1}$ for some i, since $E(\Phi^{-1}(f)) = 0$ and $E(y_4^l) = l y_4^{l-1}$. Clearly, $\mathbf{y}^d$ is equal to $y_1 y_4^{l-1}$ or $y_2 y_4^{l-1}$ or $y_3 y_4^{l-1}$. Hence, one of $\mathbf{x}^{\delta_1} x_4^{a_4-\gamma}$, $\mathbf{x}^{\delta_2} x_4^{a_4-\gamma}$, $\mathbf{x}^{\delta_3} x_4^{a_4-\gamma}$ appears in f. This is a contradiction, since these monomials are not in $K[\mathbf{x}]$. $\square$

We define linear functions $l_i : \mathbf{R}^4 \to \mathbf{R}$ for $i = 1, 2, 3$ by

$$l_i\big((a_1, a_2, a_3, a_4)\big) = a_j + a_k - (a_1 + a_2 + a_3)\xi_i, \tag{2.3}$$

where $j, k \in \{1, 2, 3\} \setminus \{i\}$ with $j \neq k$. Let $\mathcal{C}$ be the set of $a \in \mathbf{R}^4$ such that $l_i(a) \geqslant 0$ for $i = 1, 2, 3$. Then, $\mathcal{C}$ is a convex polyhedral cone in $\mathbf{R}^4$. We remark that $\mathrm{supp}(f) \subset \mathcal{C}$ implies $\Phi(f) \in K[\mathbf{x}]$ for each $f \in K[\mathbf{y}]$. To verify this, we may assume that $f = \mathbf{y}^b$ for some $b = (b_1, b_2, b_3, b_4) \in (\mathbf{Z}_{\geqslant 0})^4$. Since $b \in \mathcal{C}$, we have

$$\begin{aligned} 0 &\leqslant \big(\delta_{i,i} + \min\{\delta_{j,i}, \delta_{k,i}\}\big) l_i(b) \\ &= (b_j + b_k)\big(\delta_{i,i} + \min\{\delta_{j,i}, \delta_{k,i}\}\big) - (b_1 + b_2 + b_3)\delta_{i,i} \\ &= (b_j + b_k)\min\{\delta_{j,i}, \delta_{k,i}\} - b_i \delta_{i,i} \end{aligned} \tag{2.4}$$

for $i = 1, 2, 3$. The ith component of $a = \sum_{i=1}^3 b_i \delta_i + b_4 \gamma \mathbf{e}_4$ is not less than the right-hand side of the last equality in (2.4) for $i = 1, 2, 3$. The fourth component of a is always nonnegative. Thus, $\Phi(\mathbf{y}^b) = \mathbf{x}^a$ is in $K[\mathbf{x}]$.

The following is the key lemma.

Lemma 2.4. *There exist positive integers* p_1, p_2, *and* p_3 *with the following properties*:

(i) *The first three components of* $p_3\delta_2 + (p_1 + p_2)\delta_3$ *are positive.*
(ii) *For each positive integer* q, *there exists an element of* $K[\mathbf{y}]^E$ *of the form*

$$(y_3 - y_2)^{p_1}(y_3 - y_1)^{p_2}(y_2 - y_1)^{p_3} y_4^q + (\textit{terms of lower degree in } y_4) \tag{2.5}$$

whose support is contained in $\mathcal{C}$.

We show Theorem 1.1 as a consequence of Lemma 2.4. Let S be the union of $\mathrm{supp}(f)$ for $f \in K(\Pi) \cap K[\mathbf{x}]$. Then, the function $N : S \setminus \{0\} \to \mathbf{R}$ sending $a = (a_1, a_2, a_3, a_4) \in S \setminus \{0\}$ to $a_4/(a_1 + a_2 + a_3)$ is well-defined, since $a_1 + a_2 + a_3 = 0$ implies $a = 0$ by Lemma 2.3. Suppose to the contrary that $K(\Pi) \cap K[\mathbf{x}]$ is generated by a finite number of elements $g_1, \dots, g_r$. Then, S is contained in the subsemigroup of $\mathbf{Z}^4$ generated by $S' = \bigcup_{i=1}^{r} \mathrm{supp}(g_i)$. Since $S' \setminus \{0\}$ is a finite subset of $S \setminus \{0\}$, there exists $\mu > 0$ such that $N(a) < \mu$ for every $a \in S' \setminus \{0\}$. Note that $N(a + b) \leqslant \max\{N(a), N(b)\}$ for a and b. Thus, we have $N(a) < \mu$ for any $a \in S \setminus \{0\}$. Take $p_1, p_2, p_3 > 0$ as in Lemma 2.4. Then, there exists an element of $K[\mathbf{y}]^E$ as in (2.5) whose support is contained in $\mathcal{C}$ for each $q > 0$ by Lemma 2.4. It is sent to $f \in K(\Pi) \cap K[\mathbf{x}]$ of the form $f = f' + f''$ by Φ. Here,

$$f' = \left(\mathbf{x}^{\delta_3} - \mathbf{x}^{\delta_2}\right)^{p_1}\left(\mathbf{x}^{\delta_3} - \mathbf{x}^{\delta_1}\right)^{p_2}\left(\mathbf{x}^{\delta_2} - \mathbf{x}^{\delta_1}\right)^{p_3} x_4^{q\gamma}$$

and f'' is an element of $K[\mathbf{x}]$ with $\mathrm{supp}(f') \cap \mathrm{supp}(f'') = \emptyset$. Hence, $c = p_3\delta_2 + (p_1 + p_2)\delta_3 + q\gamma\mathbf{e}_4$ is in S for any $q > 0$. This is a contradiction, since $N(c) > \mu$ for sufficiently large q. Therefore, $K(\Pi) \cap K[\mathbf{x}]$ is not finitely generated.

Note that c is a vertex of the Newton polytope of f, i.e., the convex hull of $\mathrm{supp}(f)$ in $\mathbf{R}^4$. Moreover, every component of c is not zero by Lemma 2.4. Let D be a nonzero locally nilpotent derivation on $K[\mathbf{x}]$. Then, by Hadas and Makar-Limanov's theorem [5, Theorem 3.2], at least one component of each vertex of the Newton polytope of an element of $K[\mathbf{x}]^D \setminus \{0\}$ must be zero. Hence, $f \notin K[\mathbf{x}]^D$. Therefore, $K(\Pi) \cap K[\mathbf{x}] \not\subset K[\mathbf{x}]^D$ for any nonzero locally nilpotent derivation D on $K[\mathbf{x}]$. Thus, the proof of Theorem 1.1 is completed on the assumption that Lemma 2.4 is true.

3. Proof of Lemma 2.4

In this section, we prove Lemma 2.4. For each $r \in \mathbf{R}$, we denote by $\lfloor r \rfloor$ the minimal integer m such that $m \geqslant r$. Note that $r \leqslant \lfloor r \rfloor < r + 1$ for $r \in \mathbf{R}$.

Lemma 3.1. *There exists a positive integer p_0 with the following property. For each integer $p \geqslant p_0$, there exist positive integers p_1 and p_2 with $p_1 \geqslant p\xi_1$ such that*

$$\begin{aligned} b = {} & (p - p_2 + r)\mathbf{e}_2 + (p_2 - r)\mathbf{e}_3 + \alpha_1 p_1'(\mathbf{e}_3 - \mathbf{e}_2) + \alpha_2(p_2 - r)(\mathbf{e}_1 - \mathbf{e}_3) \\ & + \alpha_3(p - p_1' - p_2 + r)(\mathbf{e}_1 - \mathbf{e}_2) \end{aligned} \tag{3.1}$$

satisfies $l_2(b), l_3(b) \geqslant 0$ for any $\alpha_1, \alpha_2, \alpha_3 \in [0, 1]$, $0 \leqslant p_1' \leqslant p_1$, and $r = 0$. If $p_1' = p_1$, then b is contained in $\mathcal{C}$ for any $\alpha_1, \alpha_2, \alpha_3 \in [0, 1]$ and $r \in \{-1, 0, 1\}$.

Proof. By hypothesis, $\xi_1 + \xi_2 + \xi_3 < 1$. Let p_0 be a positive integer such that $p_0(1 - \xi_1 - \xi_2 - \xi_3) \geqslant 4$, and p an integer such that $p \geqslant p_0$. Set $p_1 = p - \lfloor p\xi_2 \rfloor - \lfloor p\xi_3 \rfloor - 2$ and $p_2 = \lfloor p\xi_2 \rfloor + 1$. Then, we have $p_1 > p(1 - \xi_2 - \xi_3) - 4 \geqslant p\xi_1$. For $r \in \{-1, 0, 1\}$ and $0 \leqslant p_1' \leqslant p_1$, put

$$b_1 = (p - p_2 + r)\mathbf{e}_2 + (p_2 - r)\mathbf{e}_3, \qquad b_2 = b_1 + (p_2 - r)(\mathbf{e}_1 - \mathbf{e}_3),$$
$$b_3 = b_2 + (p - p_1' - p_2 + r)(\mathbf{e}_1 - \mathbf{e}_2), \qquad b_4 = b_1 + p_1'(\mathbf{e}_3 - \mathbf{e}_2),$$
$$b_5 = b_4 + (p - p_1' - p_2 + r)(\mathbf{e}_1 - \mathbf{e}_2), \qquad b_6 = b_5 + (p_2 - r)(\mathbf{e}_1 - \mathbf{e}_3).$$

Then, b in (3.1) is contained in the convex hull of $b_1, \ldots, b_6$ in $\mathbf{R}^4$ for each $\alpha_1, \alpha_2, \alpha_3 \in [0,1]$. Hence, to complete the proof, it suffices to verify for $j = 1, \ldots, 6$ and $r \in \{-1, 0, 1\}$ that $l_2(b_j), l_3(b_j) \geqslant 0$ for $0 \leqslant p_1' \leqslant p_1$ and $l_1(b_j) \geqslant 0$ for $p_1' = p_1$.

For $0 \leqslant p_1' \leqslant p_1$ and $r \in \{-1, 0, 1\}$, we have

$$l_2(b_1) = p_2 - r - p\xi_2 = \lfloor p\xi_2 \rfloor + 1 - r - p\xi_2 \geqslant p\xi_2 + 1 - r - p\xi_2 = 1 - r \geqslant 0,$$
$$l_3(b_1) = p - \big(\lfloor p\xi_2 \rfloor + 1\big) + r - p\xi_3 > p(1 - \xi_2 - \xi_3) + r - 2 \geqslant p\xi_1 + r + 2 > 0. \quad (3.2)$$

Note that $p - p_1' - p_2 + r \geqslant p - p_1 - p_2 + r = \lfloor p\xi_3 \rfloor + 1 + r > 0$ and $p_2 - r = \lfloor p\xi_2 \rfloor - r + 1 > 0$. Moreover,

$$l_i(\mathbf{e}_3 - \mathbf{e}_2) = \begin{cases} 0 & \text{if } i = 1, \\ 1 & \text{if } i = 2, \\ -1 & \text{if } i = 3, \end{cases} \quad \text{and} \quad l_j(\mathbf{e}_1 - \mathbf{e}_k) = \begin{cases} -1 & \text{if } j = 1,\ k \in \{2,3\}, \\ 1 & \text{if } j, k \in \{2,3\},\ j = k, \\ 0 & \text{if } j, k \in \{2,3\},\ j \neq k. \end{cases}$$

Hence, $l_2(b_1) \leqslant l_2(b_4)$ and $l_i(b_1) \leqslant l_i(b_2) \leqslant l_i(b_3)$, $l_i(b_4) \leqslant l_i(b_5) \leqslant l_i(b_6)$ for $i = 2, 3$. Thus, we get $l_2(b_j), l_3(b_j) \geqslant 0$ for each j for $0 \leqslant p_1' \leqslant p_1$ by (3.2) and

$$l_3(b_4) = l_3(b_1) + p_1' l_3(\mathbf{e}_3 - \mathbf{e}_2) = l_3(b_1) - p_1' \geqslant l_3(b_1) - p_1 \geqslant r + 1 \geqslant 0.$$

Since $l_1(b_1) = l_1(b_4)$, we have $l_1(b_3) = l_1(b_6)$. Furthermore, $l_1(b_1) > l_1(b_2) > l_1(b_3)$ and $l_1(b_4) > l_1(b_5) > l_1(b_6)$. If $p_1' = p_1$, then

$$l_1(b_3) = l_1(b_1) - (p_2 - r) - (p - p_1 - p_2 + r) = -p\xi_1 + p_1 > 0.$$

Therefore, we get $l_1(b_j) \geqslant 0$ for each j if $p_1' = p_1$. $\square$

Now, take p_0 as in Lemma 3.1, and choose p_1 and p_2 for $p = p_0$. We set

$$\begin{aligned} f_r &= (y_3 - y_2)^{p_1}(y_3 - y_1)^{p_2 - r}(y_2 - y_1)^{p_3 + r} \\ &= y_2^{p - p_2 + r} y_3^{p_2 - r} \big(y_2^{-1} y_3 - 1\big)^{p_1} \big(1 - y_1 y_3^{-1}\big)^{p_2 - r} \big(1 - y_1 y_2^{-1}\big)^{p_3 + r} \end{aligned}$$

for $r \in \{-1, 0, 1\}$, where $p_3 = p - p_1 - p_2$. Then, each element of $\mathrm{supp}(f_r)$ is written as b in (3.1) with $p_1' = p_1$ for some $\alpha_1, \alpha_2, \alpha_3 \in [0,1]$. Hence, $\mathrm{supp}(f_r)$ is contained in $\mathcal{C}$ by Lemma 3.1, and so $\Phi(f_r)$ is in $K[\mathbf{x}]$ for each r. In particular, $(p_3 + r)\delta_2 + (p_1 + p_2 - r)\delta_3$ is in $(\mathbf{Z}_{\geqslant 0})^4$ for each r. This implies that the first three components of $p_3\delta_2 + (p_1 + p_2)\delta_3$ are positive. Thus, the first statement of Lemma 2.4 is proved.

We remark that, for $f' \in K[\mathbf{y}]$ and $e' \in \mathbf{Z}_{\geqslant 0}$, if $l_2(b), l_3(b) \geqslant 0$ for any $b \in \mathrm{supp}(f')$, then each of $b \in \mathrm{supp}(f'(y_4 - y_1)^{e'})$ also satisfies $l_2(b), l_3(b) \geqslant 0$. Actually, we have $l_i(\mathbf{e}_4), l_i(\mathbf{e}_1 - \mathbf{e}_4) \geqslant 0$ for $i = 2, 3$. Take any positive integer q, and let

$\mathcal{F}$ be the set of homogeneous $(p+q)$-forms $F \in K[\mathbf{y}]^E$ of the form $F = f_0 y_4^q +$ (terms of lower degree in y_4) such that $l_2(b), l_3(b) \geqslant 0$ for any $b \in \operatorname{supp}(F)$. Since $\operatorname{supp}(f_0) \subset \mathcal{C}$, we have $f_0(y_4 - y_1)^q \in \mathcal{F}$, as remarked. So, $\mathcal{F}$ is not empty. To complete the proof of the second statement of Lemma 2.4, it suffices to show the existence of $F \in \mathcal{F}$ such that $l_1(b) \geqslant 0$ for any $b \in \operatorname{supp}(F)$. Suppose the contrary, that is, there exists $b \in \operatorname{supp}(F)$ such that $l_1(b) < 0$ for every $F \in \mathcal{F}$. We define $O(F) = (d, e) \in \mathbf{Z}^2$ for each $F \in \mathcal{F}$ by

$$e = \max\bigl\{b_4 \bigm| b = (b_1, b_2, b_3, b_4) \in \operatorname{supp}(F),\ l_1(b) < 0\bigr\},$$
$$d = \max\bigl\{b_1 \bigm| b = (b_1, b_2, b_3, b_4) \in \operatorname{supp}(F),\ b_4 = e\bigr\}.$$

Consider the total order $\preccurlyeq$ on $\mathbf{Z}^2$ defined by $(d_1, e_1) \preccurlyeq (d_2, e_2)$ if $e_1 < e_2$ or $e_1 = e_2$, $d_1 \leqslant d_2$. Then, choose $F \in \mathcal{F}$ such that $O(F) \preccurlyeq O(F')$ for any $F' \in \mathcal{F}$, and let $h \in K[y_2, y_3]$ be the coefficient of $y_1^d y_4^e$ in F, where $O(F) = (d, e)$. Put $s = p + q - d - e$. If b is an element of $\operatorname{supp}(F)$ with first and fourth components d and e, respectively, then $l_1(b) = s - (p+q-e)\xi_1 = (p+q-e)(1-\xi_1) - d$. By the maximality of d, the right-hand side of this equality is negative. Hence, we get

$$s < (p + q - e)\xi_1. \tag{3.3}$$

Lemma 3.2. *There exists $\beta \in K \setminus \{0\}$ such that $h = \beta(y_3 - y_2)^s$.*

Proof. It suffices to show that $E(h) = 0$. Indeed, $K[y_2, y_3]^E = K[y_3 - y_2]$, and h is a homogeneous s-form in y_2 and y_3. Suppose to the contrary that a monomial $y_2^{b_2} y_3^{b_3}$ appears in $E(h)$ with nonzero coefficient. Then, $c - \mathbf{e}_i = (d, b_2, b_3, e)$ for some $c \in \operatorname{supp}(F)$ and $i \in \{1, 4\}$, since $E(F) = 0$. By the maximality of d, we have $i \neq 1$. Hence, $c = (d, b_2, b_3, e+1)$. Since $b_2 + b_3 = p + q - d - e - 1$ and $\xi_1 < 1$, we have

$$l_1(c) = b_2 + b_3 - \xi_1(d + b_2 + b_3) = s - (p + q - e)\xi_1 + (\xi_1 - 1) < 0$$

by (3.3). This contradicts the maximality of e. Therefore, $E(h) = 0$. $\square$

We put $\tilde{p} = p + q - e$. Then, $\tilde{p} > p$, since $0 \leqslant e < q$. So, by Lemma 3.1, there exist positive integers $\tilde{p}_1$ and $\tilde{p}_2$ with $\tilde{p}_1 \geqslant \tilde{p}\xi_1$ such that

$$\begin{aligned} b = {} & (\tilde{p} - \tilde{p}_2)\mathbf{e}_2 + \tilde{p}_2\mathbf{e}_3 + \alpha_1 p_1'(\mathbf{e}_3 - \mathbf{e}_2) + \alpha_2\tilde{p}_2(\mathbf{e}_1 - \mathbf{e}_3) \\ & + \alpha_3(\tilde{p} - p_1' - \tilde{p}_2)(\mathbf{e}_1 - \mathbf{e}_2) \end{aligned} \tag{3.4}$$

satisfies $l_2(b), l_3(b) \geqslant 0$ for any $\alpha_1, \alpha_2, \alpha_3 \in [0, 1]$ and $0 \leqslant p_1' \leqslant \tilde{p}_1$. Every element of the support of $g = (y_3 - y_2)^s (y_1 - y_3)^{\tilde{p}_2} (y_1 - y_2)^{\tilde{p} - s - \tilde{p}_2}$ is expressed as b in (3.4) with $p_1' = s$ for some $\alpha_1, \alpha_2, \alpha_3 \in [0, 1]$. Since $0 < s < \tilde{p}\xi_1 \leqslant \tilde{p}_1$ by (3.3), we get $l_2(b), l_3(b) \geqslant 0$ for any $b \in \operatorname{supp}(g)$. Hence, $l_2(b), l_3(b) \geqslant 0$ for any $b \in \operatorname{supp}(g(y_4 - y_1)^e)$, as remarked above. So, $H = F - \beta g(y_4 - y_1)^e$ is in $\mathcal{F}$. Note that $g(y_4 - y_1)^e$ is written as $(y_3 - y_2)^s y_1^d y_4^e + w$, where $w \in K[\mathbf{y}]$ such that the first and the fourth component d' and e' of each $b \in \operatorname{supp}(w)$ satisfy $(d', e') \preccurlyeq (d, e)$ and $(d', e') \neq (d, e)$. Since $h = \beta(y_3 - y_2)^s$ by Lemma 3.2, we have $O(H) \preccurlyeq O(F)$ and $O(H) \neq O(F)$. This contradicts the choice of F. Therefore, the proof of Lemma 2.4 is completed.

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